Infinity Crystals for Certain Generalized Quantum Groups

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Classical Case

Definition (Quiver)

A *quiver* is a set of vertices with directed edges between them where there may be loops at a vertex or multiple edges between a pair of vertices.

From a quiver without loops we can associate:

- an algebraic object known as a *quantum group*
- a combinatorial object called a *crystal*, known as $\mathcal{B}(\infty)$

Classical case: Previous work provides a combinatorial interpretation for $\mathcal{B}(\infty)$ associated to quantum groups of classical type corresponding to certain quivers with no loops at a vertex.

Our Project

Main Motivation: In [Bozec], a definition of a *generalized quantum group* was given, that allows for quivers with loops. We attempt find combinatorial interpretations for $\mathcal{B}(\infty)$ of these generalized quantum groups.

We work with the following quiver:

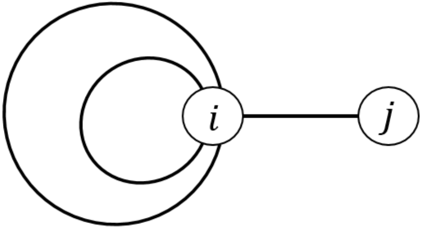

Figure: We denote this quiver as L.

Notions about Quivers

We fix a quiver Q.

Definition

If j is a vertex of Q that has no loops, then it is called a *real* vertex.

Definition

If j is a vertex of Q with more than 0 loops, then it is called an *imaginary* vertex.

Example

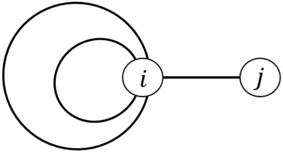

Figure: The vertex i is imaginary and the vertex j is real.

Quantum group associated to L

We study the negative portion of the quantum group, which we denote as U^-. U^- is an algebra generated by the elements $F_{i,\ell}$ for $\ell > 0$ and F_j with coefficients in $\mathbb{Q}(v)$ (rational functions of the variable v).

Example

$F_{i,1}F_{i,8} + \frac{1+v}{v^3}F_jF_{i,13}F_j + \frac{v^2}{1+v^4}F_jF_{i,2}$ is a member of U^-.

We also have some relations between elements of the quantum group:

Definition (Serre Relation)

For any $\ell > 0$, the following equality holds:

$$F_j^{(2)}2F_{i,\ell} + F_{i,\ell}F_j^{(2)} = F_jF_{i,\ell}F_j$$

Kashiwara Operators

The Kashiwara operators act on elements of U^-. There are 2 types:

- $\tilde{f}_j$ (corresponding to real vertices j)
- $\tilde{f}_{i,\ell}$ for $\ell > 0$ (corresponding to imaginary vertices i)

Example

$\tilde{f}_j(F_j) = F_j^{(2)}$
$\tilde{f}_{i,1}(F_j) = F_{i,1}F_j.$
$\tilde{f}_j(F_{i,1}F_j) = F_j(F_{i,1}F_j - v^{-1}F_jF_{i,1}) + v^{-1}F_j^{(2)}F_{i,1}$

$\mathcal{B}(\infty)$ of U^-

Definition ($\mathcal{L}(\infty)$)

$\mathcal{L}(\infty)$ is the $\mathbb{Q}(v^{-1})$ linear span of all elements of U^- that can be obtained from applying successive Kashiwara operators to 1.

Definition ($\mathcal{B}(\infty)$)

$\mathcal{B}(\infty)$ is the set of elements of U^- obtained from applying successive Kashiwara operators to 1 in the quotient of $\mathcal{L}(\infty)$ by $v^{-1}\mathcal{L}(\infty)$.

We are interested in describing when 2 sequences of Kashiwara operators applied to 1 are equal in $\mathcal{B}(\infty)$.

Example

$\tilde{f}_{i,1}\tilde{f}_j^2(1) = \tilde{f}_j\tilde{f}_{i,1}\tilde{f}_j(1)$

Kashiwara operators on $\mathcal{B}(\infty)$

Given a lattice with a j and i axis, we can associate each element of U^- with a point on the lattice based on the number of j's and i's within the element (where $F_{i,\ell}$ adds ℓ number of i's).

Example

The element $F_{i,2}F_jF_{i,3}F_j^2$ is associated to the lattice point $(3, 5)$.

As operators, $\tilde{f}_j$ moves to the right on the lattice and $\tilde{f}_{i,\ell}$ moves ℓ steps upward on the lattice.

Example

$\tilde{f}_{i,2}(F_j) = F_{i,2}F_j$ and
$\tilde{f}_j(F_{i,1}F_j) = F_j(F_{i,1}F_j - v^{-1}F_jF_{i,1}) + v^{-1}F_j^{(2)}F_{i,1}$.

We can view sequences of Kashiwara operators applied to 1 as a path on the lattice.

Conjectures

Conjecture

For any ℓ, we have the following equality in $\mathcal{B}(\infty)$:

$$\tilde{f}_{i,\ell}\tilde{f}_j^{\ell+1}(1) = \tilde{f}_j\tilde{f}_{i,\ell}\tilde{f}_j^{\ell}(1).$$

The above equality has a nice geometric interpretation in terms of equalities of lattice paths.

Conjecture

If b and b' are 2 sequences of Kashiwara operators such that $b(1) = b'(1)$, then given $k \in U^-$, $b(k) = b'(k)$.

Conjecture 2 allows us to perform the move given by Conjecture 1 anywhere along a path corresponding to a sequence of Kashiwara operators.

Combinatorial Structures

Definition (Steep path)

A steep path is a sequence of Kashiwara operators $\tilde{f}_j^{s_{m+1}} \tilde{f}_{i,t_m} \tilde{f}_j^{s_m} \dots \tilde{f}_{i,t_1} \tilde{f}_j^{s_1}$, where for all k, $t_k \geq s_k$.

From Conjecture 1 and 2, it follows that:

Claim

Any sequence of Kashiwara operators is equal to a steep path.

Claim

No two steep paths are equal.

A combinatorial characterization

There is a formula that gives the number of distinct Kashiwarwa operators applied to 1 that end at a given lattice point.

Claim

The number of steep paths to a lattice point (n, m) *is the number of distinct Kashiwara operators applied to* 1 *that end at* (n, m)*.*

Conjecture (Main Conjecture)

2 sequences of Kashiwara operators are equal on $\mathcal{B}(\infty)$ *if their corresponding steep paths are equal.*

The above theorem gives a combinatorial characterization of Kashiwara operators, as we desired.

Example

Process to decide equality of 2 sequences of Kashiwara operators:

- transform each path to its corresponding steep path
- if the steep paths are equal then the operators are equal
- if the steep paths are not equal then the operators are distinct

Combinatorial process of applying Kashiwara operators:

- add on path to existing path
- make the new path steep

Ideas and future directions

Ideas

- Conjecture 1: Use explicit description of $\tilde{f}_j$ and $\tilde{f}_{i,\ell}$ and Serre relation.
- Conjecture 2: Still thinking!
- Our proof that the number of steep paths to a lattice point is equal to the number of distinct sequences of Kashiwara operators to that point provides great evidence to support the above 2 conjectures.

Future Directions

Explore more quivers, in particular adding more real vertices to a central imaginary vertex. Our work in this case already implies many facts about these future cases.

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Bibliography

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